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Mass transport under sea waves propagating over a rippled bed

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Mass transport under a progressive sea wave propagating over a rippled bed is investigated. Wave amplitudes a^* of the same order of magnitude as that of the boundary layer thickness δ^* and of the ripple wavelength l^* are considered. All the above quantities are assumed to be much smaller than the wavelength L^* of the sea wave and much larger than the amplitude $2\epsilon^*$ of the ripples. The analysis is carried out up to the second order in the wave slope a^*/L^* and in the parameter ϵ^*/δ^* which is a measure of ripple steepness. Because of these assumptions, the slow damping of wave amplitude in the direction of wave propagation is taken into account. Attention is focused on the bottom boundary layer where an order $(\epsilon^*/\delta^*)^2$ correction of the steady velocity components described by Longuet-Higgins (1953) is found. This correction persists at the outer edge of the bottom boundary layer and affects the solution in the entire water column.

1. Introduction

The generation of steady streaming near a solid wall by an oscillating fluid is a well known phenomenon. Rayleigh (1883) first analysed the steady streaming induced by an acoustic wave in a closed duct and determined the Eulerian drift in the wall boundary layer.

The extension to sea waves propagating over a flat bed of constant depth was first made by Longuet-Higgins (1953) and since then the problem has been tackled by many researchers using various analytical, perturbative and direct numerical approaches, see for instance Riley (1965), Carter, Liu & Mei (1973), Dore (1974), Liu & Davis (1977), Haddon & Riley (1983) and the recent contributions by Iskandarani & Liu (1991) and Wen & Liu (1994). A steady velocity component is found, of second order in the wave slope, which is generated inside the bottom boundary layer and persists at the outer edge. A weak mean velocity of second order in the wave slope is also induced at the outer edge of the free-surface boundary layer. Then the residual mean vorticities at the bottom and the free surface are diffused and advected into the entire water column giving rise to a complex steady drift. The existence of steady streaming, which is directed shoreward near the bottom, is clearly relevant to questions involving the movement of sediment by wave action.

Of course the Stokes' drift is modified by the presence of bedforms. Riley (1984) determined the steady streaming induced by surface gravity waves propagating over bedforms characterized by wavelengths of the same order of magnitude as the wavelengths of the sea waves but with much larger amplitudes. Such bottom topography can be representative of sand waves or multiple bars (for a review on this topic see Mei & Liu 1993).

In the present paper we study the steady drift induced by ripples, which are small bedforms characterized by a wavelength l^* which scales with the amplitude of the oscillations of fluid displacement close to the bottom (e.g. $l^* \sim 10$ cm). Because the wavelength of ripples is many orders of magnitude smaller than that of sand waves or multiple bars and ripple amplitude is quite small, the balances in the momentum equations analysed by Riley (1984) are quite different from those characteristic of ripples and the results described in Riley (1984) are not relevant to discuss the steady drift induced by ripples.

When a rippled bed is considered, steady streaming of first order in the wave slope is found (Sleath 1984). Such steady streaming is confined in a bottom layer and consists of recirculating cells, the form, intensity and direction of which depend on the characteristics of the sea wave and of the bottom waviness. Many works have been devoted to the study of this flow because of the relevant role it plays in sediment transport and in the process of ripple formation. Indeed, because the sediment is driven by the fluid, if the steady drift in the vicinity of the bed is directed from the troughs towards the crests of the bottom waviness and is strong enough, the amplitude of the sandy bottom wave grows and ripples appear. However, since second-order effects in the wave slope have usually been neglected, the steady streaming turn out to be periodic in the direction of wave propagation and no contribution arises to the mass flux in the direction of wave propagation.

To the authors' knowledge, the only paper which describes the mass flux induced close to a wavy bed is that by Sleath (1974). However the analysis carried out by Sleath (1974) considers amplitudes of the oscillations of fluid displacement much smaller than ripple wavelength. Hence the results obtained are not relevant for active ripples but only for relic ripples, as pointed out by Sleath (1984) himself.

In the present paper the investigation of the flow over a rippled bed induced by a propagating sea wave is carried out up to order $(a^*/L^*)^2$, including terms of order $(\epsilon^*/\delta^*)^2$ and a^*, and l^* are supposed of the same order of magnitude (a^* and L^* are the amplitude and the wavelength of the sea wave respectively while ϵ^* is a measure of ripple amplitude and δ^* the thickness of the viscous bottom boundary layer). Hence two main results are obtained. At order $(a^*/L^*)^2(\epsilon^*/\delta^*)$ a distorsion of the steady recirculating cells described by Sleath (1976), Vittori (1989), Hara & Mei (1990) and Blondeaux (1990) is found: the cells are no longer symmetric with respect to ripple crests. It can be argued that this distortion induces ripple migration when a non-cohesive bottom is considered. The velocity of migration and in general the influence of $O((a^*/L^*)^2)$ terms on ripple formation and development will be described in a forthcoming paper (Blondeaux, Foti & Vittori (1996)). At order $(a^*/L^*)^2(\epsilon^*/\delta^*)^2$ a velocity component independent both of time and of x^* (the direction of wave propagation) is found. This is the major achievement of the work since it describes the correction induced by the presence of ripples on the Eulerian steady drift characteristic of the flat bottom case.

The investigation is focused in the region close to the bottom since in the core and surface regions the solution can be found by means of existing approaches.

The rest of the paper is as follows. In the next section we formulate the problem and describe the main assumptions. In §3 the solution procedure is presented and the results are discussed in §4.

2. Formulation of the problem

Let us consider the propagation of a two-dimensional surface gravity wave of length L^* and period T^* over a region of constant depth h^*. Let us denote by a^* its small amplitude. If we introduce a Cartesian coordinate system with the (x^*, z^*)-plane coincident with the bottom, the x^*-axis in the direction of wave propagation pointing offshore and the y^*-axis vertical and pointing upwards, the free-surface displacement is described by

$$y^* = h^* + \zeta^*(x^*, t^*) = h^* + \left[\tfrac{1}{2}a^*(x^*)\mathrm{e}^{\mathrm{i}(k^*x^*+\omega^*t^*)} + \text{c.c.}\right] \tag{2.1}$$

where $k^* = 2\pi/L^*$ and $\omega^* = 2\pi/T^*$ are the wavenumber and the angular frequency of the sea wave respectively. Moreover in (2.1) c.c. denotes the complex conjugate of a complex quantity. The x^* dependence of a^* is introduced because of the presence of viscous effects which cause the amplitude to decay during wave propagation.

When the flat bottom case is analysed, an extensive literature is available on the flow induced by the wave and in particular on its steady part. Here let us consider a rippled bed and denote by $\epsilon^*\eta$ the bottom elevation. For simplicity a sinusoidal profile will be considered, described by

$$y^* = \epsilon^*\eta(x^*) = \epsilon^*\mathrm{e}^{\mathrm{i}\alpha^*x^*} + \text{c.c.} \tag{2.2}$$

The problem of flow determination is posed by continuity and Navier–Stokes equations along with the kinematic and dynamic boundary conditions at the free surface and the bottom. If we introduce the dimensionless variables

$$\left.\begin{aligned} (x,y) = \frac{(x^*,y^*)}{L^*}, \quad t = t^*\omega^*, \quad \zeta = \frac{\zeta^*}{a_o^*/\sinh(2\pi h^*/L^*)}, \\ (u,v) = \frac{(u^*,v^*)}{a_o^*\omega^*/\sinh(2\pi h^*/L^*)}, \quad p = \frac{p^*}{\rho a_o^*(\omega^*)^2 L^*/\sinh(2\pi h^*/L^*)} \end{aligned}\right\} \tag{2.3}$$

(where ρ is water density, u^* and v^* are the velocity components along x^* and y^* respectively, p^* is pressure and a_o^* is a measure of the wave amplitude) the following problem is obtained:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.4}$$

$$\frac{\partial u}{\partial t} + \frac{a_o^*}{L^*\sinh(2\pi h^*/L^*)}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\nu}{\omega^*(L^*)^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2.5}$$

$$\frac{\partial v}{\partial t} + \frac{a_o^*}{L^*\sinh(2\pi h^*/L^*)}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\nu}{\omega^*(L^*)^2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{2.6}$$

$$u = v = 0 \qquad \text{for} \quad y = (\epsilon^*/L^*)\mathrm{e}^{\mathrm{i}\alpha^*L^*x} + \text{c.c.}, \tag{2.7}$$

$$\frac{\partial \zeta}{\partial t} + \frac{a_o^*}{L^*\sinh(2\pi h^*/L^*)}u\frac{\partial \zeta}{\partial x} - v = 0 \quad \text{for} \quad y = \frac{h^*}{L^*} + \frac{a_o^*}{L^*\sinh(2\pi h^*/L^*)}\zeta, \tag{2.8}$$

$$n_x^2 T_{xx} + n_y^2 T_{yy} + 2n_x n_y T_{xy} = 0 \quad \text{for} \quad y = \frac{h^*}{L^*} + \frac{a_o^*}{L^*\sinh(2\pi h^*/L^*)}\zeta, \tag{2.9}$$

$$(n_y^2 - n_x^2)T_{xy} + n_x n_y(T_{xx} - T_{yy}) = 0 \quad \text{for} \quad y = \frac{h^*}{L^*} + \frac{a_o^*}{L^*\sinh(2\pi h^*/L^*)}\zeta, \tag{2.10}$$

where ν is the kinematic viscosity of the water, n_x and n_y are the x- and y-components of the unit vector normal to the free surface and T_{ij} are the components of the stress tensor which has been scaled as pressure. Note that the effects associated with the surface tension have been neglected by assuming large values of the Weber number.

The $O(1)$ factor $\sinh(2\pi h^*/L^*)$ (hereinafter denoted by S) appearing in (2.3) has been introduced to simplify the analysis in the bottom boundary layer, where we shall focus our attention. The parameters appearing in (2.4)–(2.10) (namely $a_o^*/(SL^*)$, $\nu/(\omega^*(L^*)^2)$, h^*/L^*, ϵ^*/L^*, $\alpha^* L^*$) represent the ratios of the different length scales involved in the problem. Indeed $2\nu/\omega^*$ is the square of the thickness δ^* of the viscous bottom boundary layer.

Let us consider a small-amplitude wave propagating over intermediate depths, such that

$$a_o^* \ll L^* \sim h^*. \tag{2.11}$$

Field evidence shows that ripple wavelength l^* scales with the amplitude of fluid displacement close to the bottom, hence we have

$$l^* = 2\pi/\alpha^* \sim a_o^*/S. \tag{2.12}$$

Moreover let us assume that the amplitude ϵ^* of the ripples is much smaller than δ^*. This last assumption limits the applicability of the results in field situations, since ripple amplitude is usually larger than δ^*. Indeed such large amplitudes induce flow separation at ripple crests and the free vortex sheets roll up generating large vortex structures, the dynamics of which is highly nonlinear and can be described only by numerical methods (Longuet-Higgins 1981; Blondeaux & Vittori 1991). On the other hand the assumption $\epsilon^* \ll \delta^*$ implies that nonlinear effects are weak and the analytical treatment of the problem is greatly simplified. Hence our analysis is strictly relevant only when rolling-grain ripples are considered, since they have amplitudes so small that do not cause flow separation. Moreover the study of the problem assuming small values of ϵ^* turns out to be relevant to ripple formation and development in an explanation of the presence of a steady drift (Blondeaux *et al.* 1996). Also we feel that our results provide some qualitative information even on the flow induced by vortex ripples since all the important features of the real phenomenon (i.e. nonlinear effects related both to ripple steepness and sea wave slope) are retained. Finally, it is necessary to compare the size of l^* with that of δ^*. The assumption $l^* \ll \delta^*$ is of no practical importance since active ripples are characterized by values of l^* larger than δ^*. In the present work, the flow regime in the bottom boundary layer is assumed to be laminar, hence values of the Reynolds number $Re = a_o^*\omega^*\delta^*/S\nu$ up to 100 should be considered. In this range of Re the dimensionless wavenumber $\alpha = 2\pi\delta^*/l^*$ of ripples assumes values in the range (0.1, 0.4). For the largest values of α, a perturbative approach based on the assumption $\alpha \ll 1$ is not fully justified, hence in the following α will be assumed a free parameter. Because, as pointed out previously, for active ripples $l^* \sim a^*$, it is easy to see that Re also turns out to be a free parameter subject to the only constraint that $O(Re) \sim O(\alpha^{-1})$. Indeed the ratio a^*/l^* turns out to be equal to $\alpha Re/4\pi$ and is of order 1 for real ripples. Although in the analysis α and Re are free parameters and in principle can assume any finite value, some limits to their ranges come from the numerical procedure which is used to obtain the results. This point is discussed in §4. It is worth pointing out that the assumption $l^* \sim \delta^*$ implies $\delta^* \sim a^*$ as recently assumed by Wen & Liu (1994) in a similar context.

Because of the assumptions described above, the problem is characterized by the two independent small parameters δ and ϵ,

$$\delta = \delta^*/L^* \ll 1, \qquad \epsilon = \epsilon^*/,\delta^* \ll 1, \tag{2.13}$$

since the other parameters appearing in (2.4)–(2.10) can be related to δ and ϵ:

$$\frac{a_o^*}{SL^*} = \tfrac{1}{2}\delta Re, \qquad \frac{\nu}{\omega^*(L^*)^2} = \tfrac{1}{2}\delta^2, \qquad \alpha^* L^* = \alpha/\delta. \qquad (2.14a\text{–}c)$$

It is worth noticing that $a_o^*\omega^*/S$ is the amplitude of the irrotational velocity oscillations close to the sea bed and that the Reynolds number RE based on this length scale ($RE = a_o^*\omega/(S^2\nu)$) turns out to be equal to $Re^2/2$.

3. Solution

At this stage it is useful to take into account that the amplitude of the sea wave decays on a spatial scale L^*/δ due to viscous effects (Mei 1989). Hence let us introduce the new variable

$$\chi = x\delta \qquad (3.1)$$

and assume that

$$a = \frac{a^*}{a_o^*} = a(\chi) \qquad (3.2)$$

where the function $a(\chi)$ can be set equal to 1 without loss of generality, if a region of order L^* is considered. The introduction of the slow spatial scale χ forces the derivatives in the x-direction in (2.4)–(2.10) to become $\partial/\partial x + \delta\partial/\partial\chi$

Then the flow is expanded in a power series of δ (δ is related to the wave slope because of (2.14a))

$$(u, v, p, \zeta) = (U_0, V_0, P_0, \zeta_0) + \delta(U_1, V_1, P_1, \zeta_1) + O(\delta^2). \qquad (3.3)$$

At the leading order of approximation the following solution is obtained:

$$(U_0, V_0, P_0, \zeta_0) = \left(-\frac{a}{2}\cosh(2\pi y), \frac{\mathrm{i}a}{2}\sinh(2\pi y), \frac{a}{4\pi}\cosh(2\pi y), \frac{Sa}{2}\right)\mathrm{e}^{\mathrm{i}(2\pi x+t)} + \text{c.c.} \qquad (3.4)$$

which however does not satisfy all the boundary conditions at the bottom. The existence of a bottom boundary layer is thus inferred. Let us focus our attention on this layer where the problem can be rescaled using the viscous length scale δ^*:

$$\tilde{x} = x/\delta, \qquad \tilde{y} = y/\delta. \qquad (3.5)$$

The algebra which is needed to find the solution can be considerably simplified by introducing the variable

$$\tilde{t} = t + 2\pi\delta\tilde{x} \qquad (3.6)$$

and by splitting the pressure field into a component of order δ and a component of order 1 which is forced by the external irrotational flow at the bottom and does not depend on $\tilde{y}$ and $\tilde{x}$ but only on $\tilde{t}$ and χ:

$$p(\tilde{x}, \tilde{y}, \tilde{t}, \chi) = P_e(\tilde{t}, \chi) + \delta\tilde{p}(\tilde{x}, \tilde{y}, \tilde{t}, \chi). \qquad (3.7)$$

Since

$$\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial \tilde{t}}, \qquad \frac{\partial}{\partial x} \equiv \frac{1}{\delta}\frac{\partial}{\partial \tilde{x}} + 2\pi\frac{\partial}{\partial \tilde{t}}, \qquad \frac{\partial}{\partial y} \equiv \frac{1}{\delta}\frac{\partial}{\partial \tilde{y}}, \qquad (3.8)$$

the problem in the bottom boundary layer turns out to be

$$\frac{\partial u}{\partial \tilde{x}} + \frac{\partial v}{\partial \tilde{y}} + 2\pi\delta\frac{\partial u}{\partial \tilde{t}} = O(\delta^2), \qquad (3.9)$$

$$\frac{\partial u}{\partial \tilde{t}} + \frac{Re}{2}\left[u\left(\frac{\partial u}{\partial \tilde{x}} + 2\pi\delta\frac{\partial u}{\partial \tilde{t}}\right) + v\frac{\partial u}{\partial \tilde{y}}\right] = -2\pi\frac{\partial P_e}{\partial \tilde{t}} - \delta\frac{\partial P_e}{\partial \chi} - \frac{\partial \tilde{p}}{\partial \tilde{x}} - 2\pi\delta\frac{\partial \tilde{p}}{\partial \tilde{t}}$$
$$+\frac{1}{2}\left[\frac{\partial^2 u}{\partial \tilde{x}^2} + 2\pi\delta\frac{\partial}{\partial \tilde{t}}\left(2\frac{\partial u}{\partial \tilde{x}} + 2\pi\delta\frac{\partial u}{\partial \tilde{t}}\right) + \frac{\partial^2 u}{\partial \tilde{y}^2}\right] + \text{ other terms of } O(\delta^2), \quad (3.10)$$

$$\frac{\partial v}{\partial \tilde{t}} + \frac{Re}{2}\left[u\left(\frac{\partial v}{\partial \tilde{x}} + 2\pi\delta\frac{\partial v}{\partial \tilde{t}}\right) + v\frac{\partial v}{\partial \tilde{y}}\right]$$
$$= -\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{1}{2}\left[\frac{\partial^2 v}{\partial \tilde{x}^2} + 2\pi\delta\frac{\partial}{\partial \tilde{t}}\left(2\frac{\partial v}{\partial \tilde{x}} + 2\pi\delta\frac{\partial v}{\partial \tilde{t}}\right) + \frac{\partial^2 v}{\partial \tilde{y}^2}\right] + \text{ other terms of } O(\delta^2), \quad (3.11)$$

$$u = v = 0 \qquad \text{for } \tilde{y} = \epsilon e^{i\alpha\tilde{x}} + \text{c.c.} \quad (3.12)$$

plus matching conditions between the solution of the problem (3.9)–(3.12) for $\tilde{y}$ tending to infinity and the solution (3.3) in the core region for y tending to zero. The terms of $O(\delta^2)$ not explicity written in (3.9)–(3.11) come from the dependence of the flow in the boundary layer on the slow spatial variable χ.

Since the outer solution is obtained as a power series of δ, let us expand $(u, v, P_e, \tilde{p})$ in a similar way:

$$(u, v, P_e, \tilde{p}) = (u_0, v_0, P_{e0}, \tilde{p}_0) + \delta(u_1, v_1, P_{e1}, \tilde{p}_1) + O(\delta^2). \quad (3.13)$$

In the following, capital letters indicate dependent variables in the core region where y^* scales with h^* and small letters are used for the corresponding quantities in the bottom boundary layer where lengths are scaled with δ^*.

At the leading order, the problem for $(u_0, v_0, P_{e0}, \tilde{p}_0)$ turns out to be equal to that posed by (3.9)–(3.12) without the terms of order δ. Because of the presence of the small parameter ϵ, we expand the solution as

$$(u_0, v_0, P_{e0}, \tilde{p}_0) = (u_{00}, v_{00}, P_{e0}, p_{00}) + \epsilon(u_{01}, v_{01}, 0, p_{01}) + \epsilon^2(u_{02}, v_{02}, 0, p_{02}) + O(\epsilon^3) \quad (3.14)$$

where no contribution of $O(\epsilon)$ or $O(\epsilon^2)$ is present in P_{e0}. At order ϵ^0, because the outer flow for vanishing y tends to

$$(U, V) \to (-\tfrac{1}{2}e^{i\tilde{t}} + \text{c.c.}, 0) + O(\delta) \quad (3.15)$$

and the boundary condition (3.12) for (u_{00}, v_{00}) forces

$$u_{00} = v_{00} = 0 \qquad \text{at } \tilde{y} = 0 \quad (3.16)$$

u_{00} turns out to be independent of $\tilde{x}$ and v_{00} vanishes. Moreover the continuity of the pressure field suggests

$$P_{e0} = \frac{1}{4\pi}e^{i\tilde{t}} + \text{c.c.} \quad (3.17)$$

It turns out that

$$u_{00} = \hat{u}_{00}(\tilde{y})e^{i\tilde{t}} + \text{c.c.} = -\tfrac{1}{2}[1 - e^{-(1+i)\tilde{y}}]e^{i\tilde{t}} + \text{c.c.}, \quad (3.18)$$

$$v_{00} \equiv 0, \quad (3.19)$$

$$p_{00} \equiv 0. \quad (3.20)$$

At order ϵ the following problem is obtained:

$$\frac{\partial u_{01}}{\partial \tilde{x}} + \frac{\partial v_{01}}{\partial \tilde{y}} = 0, \quad (3.21)$$

$$\frac{\partial u_{01}}{\partial \tilde{t}} + \frac{Re}{2}\left(u_{00}\frac{\partial u_{01}}{\partial \tilde{x}} + v_{01}\frac{\partial u_{00}}{\partial \tilde{y}}\right) = -\frac{\partial p_{01}}{\partial \tilde{x}} + \frac{1}{2}\left(\frac{\partial^2 u_{01}}{\partial \tilde{x}^2} + \frac{\partial^2 u_{01}}{\partial \tilde{y}^2}\right), \tag{3.22}$$

$$\frac{\partial v_{01}}{\partial \tilde{t}} + \frac{Re}{2}u_{00}\frac{\partial v_{01}}{\partial \tilde{x}} = -\frac{\partial p_{01}}{\partial \tilde{y}} + \frac{1}{2}\left(\frac{\partial^2 v_{01}}{\partial \tilde{x}^2} + \frac{\partial^2 v_{01}}{\partial \tilde{y}^2}\right), \tag{3.23}$$

$$u_{01} = -\frac{\partial u_{00}}{\partial \tilde{y}}\mathrm{e}^{\mathrm{i}\alpha\tilde{x}} + \text{c.c.} \quad \text{for } \tilde{y} = 0, \tag{3.24}$$

$$v_{01} = 0 \qquad \text{for } \tilde{y} = 0, \tag{3.25}$$

$$(u_{01}, v_{01}) \to 0 \qquad \text{as } \tilde{y} \to \infty. \tag{3.26}$$

Equations (3.21)–(3.23) along with boundary conditions (3.24)–(3.26) are those solved by Vittori (1989) and Blondeaux (1990) and similar to those considered by Hara & Mei (1990). The results show the existence of recirculating cells periodic both in time and in the direction of wave propagation. Indeed the solution can be written in the form

$$(u_{01}, v_{01}, p_{01}) = (\hat{u}_{01}(\tilde{y},\tilde{t}), \hat{v}_{01}(\tilde{y},\tilde{t}), \hat{p}_{01}(\tilde{y},\tilde{t}))\mathrm{e}^{\mathrm{i}\alpha\tilde{x}} + \text{c.c.}$$
$$= \left(\sum_{m=-\infty}^{\infty}(u_{01}^{(m)}(\tilde{y}), v_{01}^{(m)}(\tilde{y}), p_{01}^{(m)}(\tilde{y}))\mathrm{e}^{\mathrm{i}m\tilde{t}}\right)\mathrm{e}^{\mathrm{i}\alpha\tilde{x}} + \text{c.c.} \tag{3.27}$$

The existence of steady streaming can be inferred by the non-vanishing values of $u_{01}^{(0)}$, $v_{01}^{(0)}$, $p_{01}^{(0)}$. The discussion of the flow patterns when different values of Re and α are considered is not relevant here because no steady flow averaged in the $\tilde{x}$-direction is induced. The interested reader is referred to the papers mentioned above for a discussion of the results.

At order ϵ^2 the solution is characterized by a component proportional to $\mathrm{e}^{2\mathrm{i}\alpha\tilde{x}}$ and one which does not depend on $\tilde{x}$:

$$(u_{02}, v_{02}, p_{02}) = [(\hat{u}_{02}^{(2)}(\tilde{y},\tilde{t}), \hat{v}_{02}^{(2)}(\tilde{y},\tilde{t}),\ \hat{p}_{02}^{(2)}(\tilde{y},\tilde{t}))\mathrm{e}^{2\mathrm{i}\alpha\tilde{x}} + \text{c.c.}]$$
$$+ (u_{02}^{(0)}(\tilde{y},\tilde{t}), v_{02}^{(0)}(\tilde{y},\tilde{t}), p_{02}^{(0)}(\tilde{y},\tilde{t})). \tag{3.28}$$

Only the latter turns out to be relevant to evaluate the correction of order ϵ^2 of the steady drift induced by ripples. After some tedious but straightforward algebra it is possible to obtain

$$\frac{\partial v_{02}^{(0)}}{\partial \tilde{y}} = 0, \tag{3.29}$$

$$\frac{\partial u_{02}^{(0)}}{\partial \tilde{t}} + \frac{Re}{2}\left[\left(\hat{u}_{01}(\mathrm{i}\alpha\hat{u}_{01})^\dagger + \hat{v}_{01}\left(\frac{\partial \hat{u}_{01}}{\partial \tilde{y}}\right)^\dagger + v_{02}^{(0)}\frac{\partial u_{00}}{\partial \tilde{y}}\right) + \text{c.c.}\right] = \frac{1}{2}\frac{\partial^2 u_{02}^{(0)}}{\partial \tilde{y}^2}. \tag{3.30}$$

$$\frac{\partial v_{02}^{(0)}}{\partial t} + \frac{Re}{2}\left[\left(\hat{u}_{01}(\mathrm{i}\alpha\hat{v}_{01})^\dagger + \hat{v}_{01}\left(\frac{\partial \hat{v}_{01}}{\partial \tilde{y}}\right)^\dagger\right) + \text{c.c.}\right] = -\frac{\partial p_{02}^{(0)}}{\partial \tilde{y}} + \frac{1}{2}\frac{\partial^2 v_{02}^{(0)}}{\partial \tilde{y}^2}, \tag{3.31}$$

where † indicates the complex conjugate. Since the boundary condition for $v_{02}^{(0)}$ at $\tilde{y} = 0$ is

$$v_{02}^{(0)} = -\frac{\partial \hat{v}_{01}}{\partial y} + \text{c.c.} \quad \text{for } \tilde{y} = 0 \tag{3.32}$$

it forces $v_{02}^{(0)}$ to be zero at $\tilde{y} = 0$. Indeed $\partial\hat{v}_{01}/\partial\tilde{y}$ at $\tilde{y} = 0$ turns out to be equal to $\mathrm{i}\alpha(\partial u_{00}/\partial\tilde{y})$ at $\tilde{y} = 0$ and hence imaginary. Then equation (3.29) suggests that $v_{02}^{(0)} \equiv 0$.

Equation (3.30) can be solved with the approach used by Vittori (1989). Finally the pressure field can be easily obtained by means of the numerical quadrature of (3.31). It is worth pointing that equations (3.30), (3.31) are subject to the following boundary conditions:

$$u_{02}^{(0)} = \left(-\frac{\partial \hat{u}_{01}}{\partial \tilde{y}} + \text{c.c.}\right) - \frac{\partial^2 u_{00}}{\partial \tilde{y}^2} \qquad \text{for } \tilde{y} = 0, \tag{3.33}$$

$$(u_{02}^{(0)}, p_{02}^{(0)}) \to 0 \quad \text{as } \tilde{y} \to \infty. \tag{3.34}$$

At order δ, let us again distinguish between the core region where $O(y) \sim 1$ and the inner region where $O(\tilde{y}) \sim 1$. In the outer region the governing equations are

$$\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y} + \frac{\partial U_0}{\partial \chi} = 0, \tag{3.35}$$

$$\frac{\partial U_1}{\partial t} + \frac{Re}{2}\left(U_0 \frac{\partial U_0}{\partial x} + V_0 \frac{\partial U_0}{\partial y}\right) = -\frac{\partial P_1}{\partial x} - \frac{\partial P_0}{\partial \chi}, \tag{3.36}$$

$$\frac{\partial V_1}{\partial t} + \frac{Re}{2}\left(U_0 \frac{\partial V_0}{\partial x} + V_0 \frac{\partial V_0}{\partial y}\right) = -\frac{\partial P_1}{\partial y}. \tag{3.37}$$

Indeed, even though the amplitude of the basic wave can be assumed equal to 1 at the leading order of approximation in a region of order L^*, its slow variation should be taken into account when considering the $O(\delta)$ problem.

Because of the linear and nonlinear forcing terms appearing in (3.35)–(3.37) and because of the matching with the inner solution, the external solution should contain a steady part and a periodic part proportional both to $e^{i(2\pi x+t)}$ and $e^{i2(2\pi x+t)}$:

$$(U_1, V_1, P_1) = (U_{1s}, V_{1s}, P_{1s}) + [(U_{1p}^{(1)}, V_{1p}^{(1)}, P_{1p}^{(1)})e^{i(2\pi x+t)} + \text{c.c}] + [(U_{1p}^{(2)}, V_{1p}^{(2)}, P_{1p}^{(2)})e^{i2(2\pi x+t)} + \text{c.c.}]. \tag{3.38}$$

Only the periodic contributions are relevant to determining the structure of the steady flow in the bottom boundary layer.

At this stage it is worth pointing out that (U_{1s}, V_{1s}, P_{1s}) has two components: one of $O(1)$ which is equal to that described by Longuet-Higgins (1953) and one of order ϵ^2 which can be found with a similar approach.

The solution forced by the nonlinear terms appearing in (3.36), (3.37) is found to be

$$(U_{1p}^{(2)}, V_{1p}^{(2)}, P_{1p}^{(2)}) = \left(2iC_1 \cosh(4\pi y), 2C_1 \sinh(4\pi y), -i\frac{C_1}{\pi}\cosh(4\pi y) - \frac{Re}{16}\right) a^2 \tag{3.39}$$

where

$$C_1 = \frac{3}{16}\pi i Re \left[\cosh(4\pi h^*/L^*) - \frac{\sinh(4\pi h^*/L^*)}{2\tanh(2\pi h^*/L^*)}\right]^{-1}. \tag{3.40}$$

The part of the solution proportional to $e^{i(2\pi x+t)}$ is forced both by the terms $\partial U_0/\partial \chi$, $\partial P_0/\partial \chi$ appearing in (3.35), (3.36) and by the matching with the inner solution. The former contribution is

$$(U_{1p}^{(11)}, V_{1p}^{(11)}, P_{1p}^{(11)}) = \frac{1}{2}\frac{da}{d\chi}\cosh(2\pi y)\left(iy\tanh(2\pi y), y, \frac{i}{2\pi}[1/(2\pi) - y\tanh(2\pi y)]\right) \tag{3.41}$$

while the latter contribution $(U_{1p}^{(12)}, V_{1p}^{(12)}, P_{1p}^{(12)})$ is forced by the non-vanishing value of v_{10} for $\tilde{y}$ tending to infinity. By considering the $O(\delta)$ solution (3.51) in the bottom boundary layer (which will be obtained in the following) and by verifying that the term $\pi i\tilde{y}$ matches V_o for $y \to 0$, it can be seen that

$$(U_{1p}^{(12)}, V_{1p}^{(12)}, P_{1p}^{(12)}) = \left(\frac{\pi}{2}(1-\mathrm{i})\sinh(2\pi y), -\frac{\pi}{2}(1+\mathrm{i})\cosh(2\pi y), -\tfrac{1}{4}(1-\mathrm{i})\sinh(2\pi y)\right) a. \tag{3.42}$$

While the solution (3.39) satisfies both the kinematic boundary condition at the free surface and that which involves the pressure field, the component $(U_{1p}^{(1)}, V_{1p}^{(1)}, P_{1p}^{(1)})$ gives rise to a value of $\partial P_1/\partial t - V_1 \cosh[2\pi h]/(2\pi \sinh[2\pi h])$ at $y = h$ which is equal to

$$\left[\frac{\partial P_1}{\partial t} - \frac{V_1}{2\pi \sinh[2\pi h]/\cosh[2\pi h]}\right]_{y=h} = \frac{1}{4\pi}\frac{\mathrm{d}a}{\mathrm{d}\chi}\left[h \sinh(2\pi h) - \frac{\cosh(2\pi h)}{2\pi}\right]$$
$$-\frac{(1+\mathrm{i})\sinh(2\pi h)}{4}a - \frac{1}{4\pi}\frac{\mathrm{d}a}{\mathrm{d}\chi}h\frac{\cosh^2(2\pi h)}{\sinh(2\pi h)} + \tfrac{1}{4}(1+\mathrm{i})\frac{\cosh^2(2\pi h)a}{\sinh(2\pi h)} \tag{3.43}$$

and it is different from zero unless

$$\frac{\mathrm{d}a}{\mathrm{d}\chi} = \frac{2\pi^2(1+\mathrm{i})}{2\pi h + \sinh[2\pi h]\cosh[2\pi h]}a. \tag{3.44}$$

It is easy to verify that the solvability condition (3.44) gives rise to the wave amplitude decay induced by energy dissipation in the bottom boundary layer. In fact the computation of the dissipation factor f_e defined by Jonsson (1963) leads to the well-known value

$$f_e = \frac{3\pi}{4Re}. \tag{3.45}$$

At this stage it is worth pointing out that the increase of energy dissipation caused by the bottom waviness would induce a further term in (3.44) of order ϵ^2. However this term is not explicitly considered here because it does not affect the solution up to the order considered in the present work.

As for the previous order, the solution (3.38) does not satisfy the boundary conditions which involve the viscosity and the bottom boundary layer should be explicitly considered.

Once more the presence of the small parameter ϵ suggests expanding the functions $(u_1, v_1, P_{e1}, \tilde{p}_1)$ appearing in (3.13) in power series of ϵ:

$$(u_1, v_1, P_{e1}, \tilde{p}_1) = (u_{10}, v_{10}, P_{e1}, p_{10}) + \epsilon(u_{11}, v_{11}, 0, p_{11}) + \epsilon^2(u_{12}, v_{12}, 0, p_{12}) + O(\epsilon^3). \tag{3.46}$$

At the leading order, because the external forcing does not depend on the fast spatial variable $\tilde{x}$ and the no-slip condition (3.12) is forced at $\tilde{y} = 0$, it can be assumed that $(u_{10}, v_{10}, P_{e1}, p_{10})$ do not depend on $\tilde{x}$. After some algebra it is possible to obtain the following problem:

$$2\pi\frac{\partial u_{00}}{\partial \tilde{t}} + \frac{\partial v_{10}}{\partial \tilde{y}} = 0, \tag{3.47}$$

$$\frac{\partial u_{10}}{\partial \tilde{t}} + \frac{Re}{2}\left(2\pi u_{00}\frac{\partial u_{00}}{\partial \tilde{t}} + v_{10}\frac{\partial u_{00}}{\partial \tilde{y}}\right) = -2\pi\frac{\mathrm{d}P_{e1}}{\mathrm{d}t} + \frac{1}{2}\frac{\partial^2 u_{10}}{\partial \tilde{y}^2}, \tag{3.48}$$

$$\frac{\partial v_{10}}{\partial \tilde{t}} = -\frac{\partial p_{10}}{\partial \tilde{y}} + \frac{1}{2}\frac{\partial^2 v_{10}}{\partial \tilde{y}^2}, \tag{3.49}$$

$$u_{10} = v_{10} = 0 \qquad \text{for } \tilde{y} = 0, \tag{3.50}$$

where appropriate matching conditions with the external flow should be forced.

The continuity equation gives

$$v_{10} = \hat{v}_{10}(y)e^{i\tilde{t}} + \text{c.c.} = \pi i \left[\tilde{y} + \frac{1}{1+i}(e^{-(1+i)\tilde{y}} - 1)\right] e^{i\tilde{t}} + \text{c.c.} \tag{3.51}$$

while equation (3.49) allows p_{10} to be determined:

$$p_{10} = \pi \left(\frac{\tilde{y}^2}{2} - \frac{\tilde{y}}{1+i}\right) e^{i\tilde{t}} + \text{c.c.} \tag{3.52}$$

The matching condition with the external flow suggests the following expression for P_{e1}:

$$P_{e1} = -\left(\frac{iC_1}{\pi} + \frac{Re}{16}\right) e^{2i\tilde{t}} + \text{c.c.} \tag{3.53}$$

which can be recast as

$$-2\pi \frac{dP_{e1}}{dt} = \left[\frac{\partial U_{1p}^{(2)}}{\partial t} + \frac{Re}{2} U_0 \frac{\partial U_0}{\partial x}\right]_{y=0} = \left(-4C_1 + \frac{i\pi Re}{4}\right) e^{2i\tilde{t}} + \text{c.c.} \tag{3.54}$$

Hence equation (3.48) can be written as

$$\frac{\partial u_{10}}{\partial \tilde{t}} + \frac{Re}{2}\left(2\pi u_{00}\frac{\partial u_{00}}{\partial \tilde{t}} + v_{10}\frac{\partial u_{00}}{\partial \tilde{y}}\right) = \left[\frac{\partial U_{1p}^{(2)}}{\partial t} + \frac{Re}{2} U_0 \frac{\partial U_0}{\partial x}\right]_{y=0} + \frac{1}{2}\frac{\partial^2 u_{10}}{\partial \tilde{y}^2}. \tag{3.55}$$

Because of the forcing terms, u_{10} can be split into a steady component and an oscillating one:

$$u_{10} = [\bar{u}_{10}(\tilde{y}) + \hat{u}_{10}(\tilde{y})e^{2i\tilde{t}}] + \text{c.c.} \tag{3.56}$$

By substituting (3.56) in (3.55) and solving the problems for $\bar{u}_{10}$ and $\hat{u}_{10}$ it is easy to see that

$$\bar{u}_{10} = -\frac{\pi Re}{4}\left\{\tfrac{3}{2} + \tfrac{1}{2}e^{-2\tilde{y}} - e^{-(1-i)\tilde{y}}[2 + i + (1-i)\tilde{y}]\right\}, \tag{3.57}$$

$$\hat{u}_{10} = 2iC_1 - \frac{\pi Re}{4}e^{-(1+i)\tilde{y}}[1 - (1+i)\tilde{y}] + e^{-\sqrt{2}(1+i)\tilde{y}}\left[\frac{\pi Re}{4} - 2iC_1\right]. \tag{3.58}$$

Next the problem at order $\delta\epsilon$ should be solved. The algebra is tedious but straightforward and the following equations are obtained:

$$\frac{\partial u_{11}}{\partial \tilde{x}} + 2\pi\frac{\partial u_{01}}{\partial \tilde{t}} + \frac{\partial v_{11}}{\partial \tilde{y}} = 0, \tag{3.59}$$

$$\begin{aligned}\frac{\partial u_{11}}{\partial \tilde{t}} + \frac{Re}{2}\Bigg[u_{10}\frac{\partial u_{01}}{\partial \tilde{x}} + u_{00}\frac{\partial u_{11}}{\partial \tilde{x}} + 2\pi\left(u_{00}\frac{\partial u_{01}}{\partial \tilde{t}} + u_{01}\frac{\partial u_{00}}{\partial \tilde{t}}\right) + v_{10}\frac{\partial u_{01}}{\partial \tilde{y}}\\ + v_{11}\frac{\partial u_{00}}{\partial \tilde{y}} + v_{01}\frac{\partial u_{10}}{\partial \tilde{y}}\Bigg] = -\frac{\partial p_{11}}{\partial \tilde{x}} - 2\pi\frac{\partial p_{01}}{\partial \tilde{t}}\\ + \frac{1}{2}\left[\frac{\partial^2 u_{11}}{\partial \tilde{x}^2} + 4\pi\frac{\partial^2 u_{01}}{\partial \tilde{x}\partial \tilde{t}} + \frac{\partial^2 u_{11}}{\partial \tilde{y}^2}\right],\end{aligned} \tag{3.60}$$

$$\begin{aligned}\frac{\partial v_{11}}{\partial \tilde{t}} + \frac{Re}{2}\left[u_{10}\frac{v_{01}}{\partial \tilde{x}} + u_{00}\frac{\partial v_{11}}{\partial \tilde{x}} + 2\pi u_{00}\frac{\partial v_{01}}{\partial \tilde{t}} + v_{10}\frac{\partial v_{01}}{\partial \tilde{y}} + v_{01}\frac{\partial v_{10}}{\partial \tilde{y}}\right]\\ = -\frac{\partial p_{11}}{\partial \tilde{y}} + \frac{1}{2}\left[\frac{\partial^2 v_{11}}{\partial \tilde{x}^2} + 4\pi\frac{\partial^2 v_{01}}{\partial \tilde{x}\partial \tilde{t}} + \frac{\partial^2 v_{11}}{\partial \tilde{y}^2}\right],\end{aligned} \tag{3.61}$$

along with the boundary and matching conditions

$$(u_{11}, v_{11}) = -\frac{\partial(u_{10}, v_{10})}{\partial \tilde{y}} \mathrm{e}^{\mathrm{i}\alpha\tilde{x}} + \text{c.c.} \qquad \text{for } \tilde{y} = 0, \tag{3.62}$$

$$(u_{11}, v_{11}) \to 0 \qquad \text{as } \tilde{y} \to \infty. \tag{3.63}$$

Equations (3.60), (3.61) along with (3.59) can be combined to provide a single equation for v_{11}, which turns out to be

$$\begin{aligned}
&\frac{\partial}{\partial \tilde{t}}(\tilde{\nabla}^2 v_{11}) + \frac{Re}{2}\left[u_{00}\frac{\partial}{\partial \tilde{x}}(\tilde{\nabla}^2 v_{11}) - \frac{\partial^2 u_{00}}{\partial \tilde{y}^2}\frac{\partial v_{11}}{\partial \tilde{x}}\right] - \tfrac{1}{2}\tilde{\nabla}^4 v_{11} \\
&\quad = \frac{Re}{2}\left\{\frac{\partial^2}{\partial \tilde{x}\partial \tilde{y}}\left[4\pi u_{00}\frac{\partial u_{01}}{\partial \tilde{t}} + u_{10}\frac{\partial u_{01}}{\partial \tilde{x}} + 2\pi u_{01}\frac{\partial u_{00}}{\partial \tilde{t}} + v_{10}\frac{\partial u_{01}}{\partial y} + v_{01}\frac{\partial u_{10}}{\partial y}\right]\right. \\
&\qquad \left. - \frac{\partial^2}{\partial \tilde{x}^2}\left[u_{10}\frac{\partial v_{01}}{\partial \tilde{x}} + 2\pi u_{00}\frac{\partial v_{01}}{\partial \tilde{y}} + v_{01}\frac{\partial v_{10}}{\partial \tilde{y}}\right]\right\} \\
&\qquad + 2\pi\left\{\frac{1}{2}\frac{\partial^2}{\partial \tilde{t}\partial \tilde{y}}(\nabla^2 u_{01}) - \frac{\partial^4 u_{01}}{\partial \tilde{x}^2\partial \tilde{y}\partial t} - \frac{\partial^3 u_{01}}{\partial \tilde{y}\partial \tilde{t}^2} + \frac{\partial^4 v_{01}}{\partial \tilde{x}^3\partial \tilde{t}} + \frac{\partial^3 p_{01}}{\partial \tilde{x}\partial \tilde{y}\partial \tilde{t}}\right\}
\end{aligned} \tag{3.64}$$

where

$$\tilde{\nabla}^2 \equiv \frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial^2}{\partial \tilde{y}^2}, \tag{3.65}$$

with the boundary and matching conditions derived by (3.62), (3.63) using continuity equation (3.59):

$$v_{11} = -\frac{\partial v_{10}}{\partial \tilde{y}}\mathrm{e}^{\mathrm{i}\alpha\tilde{x}} + \text{c.c.}, \tag{3.66}$$

$$\frac{\partial v_{11}}{\partial \tilde{y}} = -2\pi\frac{\partial u_{01}}{\partial \tilde{t}} + \left(\mathrm{i}\alpha\frac{\partial u_{10}}{\partial \tilde{y}}\mathrm{e}^{\mathrm{i}\alpha\tilde{x}} + \text{c.c.}\right) \qquad \text{for } \tilde{y} = 0, \tag{3.67}$$

$$\left(v_{11}, \frac{\partial v_{11}}{\partial \tilde{y}}\right) \to (0, 0) \qquad \text{as } \tilde{y} \to \infty. \tag{3.68}$$

The boundary conditions (3.66) and (3.67) suggest the following form for the solution:

$$(u_{11}, v_{11}, p_{11}) = (\hat{u}_{11}(\tilde{y}, \tilde{t}), \hat{v}_{11}(\tilde{y}, \tilde{t}), \hat{p}_{11}(\tilde{y}, \tilde{t}))\mathrm{e}^{\mathrm{i}\alpha\tilde{x}} + \text{c.c.} \tag{3.69}$$

Substituting (3.69) in (3.64), a new problem is obtained where $\partial/\partial\tilde{x}$ is replaced by $\mathrm{i}\alpha$. The problem turns out to be similar to that solved by Vittori (1989). Therefore the same method of solution is employed here (the reader interested in a description of the approach is referred to the Appendix).

Once equation (3.64) is solved, u_{11} can be obtained from (3.59) which gives

$$\hat{u}^{11} = \left[-\frac{\partial \hat{v}_{11}}{\partial \tilde{y}} - 2\pi\frac{\partial \hat{u}_{01}}{\partial \tilde{t}}\right] / (\mathrm{i}\alpha) \tag{3.70}$$

and $\hat{p}_{11}$ from (3.60).

Finally, the problem of order $\delta\epsilon^2$ should be tackled. At this order of approximation the velocity field shows a component periodic in the $\tilde{x}$-direction as well as a part which does not depend on $\tilde{x}$. Let us focus our attention on the latter contribution and denote it with an index $^{(0)}$. For sake of brevity we skip the detail and give only the final equations.

From the continuity equation

$$\frac{\partial v_{12}^{(0)}}{\partial y} = -2\pi \frac{\partial u_{02}^{(0)}}{\partial \tilde{t}} \tag{3.71}$$

the vertical component of the velocity can be easily obtained by a numerical quadrature, when the boundary condition

$$v_{12}^{(0)} = -\left[\frac{\partial^2 v_{10}}{\partial \tilde{y}^2} + \left(\frac{\partial \hat{v}_{11}}{\partial \tilde{y}} + \text{c.c.}\right)\right] \quad \text{at } \tilde{y} = 0 \tag{3.72}$$

is taken into account. The momentum equation for the time average $\overline{u}_{12}^{(0)}$ of $u_{12}^{(0)}$ gives

$$\frac{\partial^2 \overline{u}_{12}^{(0)}}{\partial \tilde{y}^2} = Re\left[\overline{\left(\hat{v}_{11}\left(\frac{\partial \hat{u}_{01}}{\partial \tilde{y}}\right)^{\dagger} + \hat{v}_{01}\left(\frac{\partial \hat{u}_{11}}{\partial \tilde{y}}\right)^{\dagger}\right) + \text{c.c.} + v_{12}^{(0)}\frac{\partial u_{00}}{\partial \tilde{y}} + v_{10}\frac{\partial u_{02}^{(0)}}{\partial \tilde{y}}}\right] = \mathscr{F}(y) \tag{3.73}$$

where an overbar denote the time average. Equation (3.73) is subject to the condition

$$\overline{u}_{12}^{(0)} = -\left[\overline{\frac{\partial^2 u_{10}}{\partial \tilde{y}^2} + \left(\frac{\partial \hat{u}_{11}}{\partial y} + \text{c.c.}\right)}\right] \quad \text{at } \tilde{y} = 0; \tag{3.74}$$

moreover the matching with the external flow requires $\overline{u}_{12}^{(0)}$ to be finite for $\tilde{y}$ tending to infinity. The particular solution $\overline{u}_{12p}^{(0)}$ of (3.73) can be determined with a numerical quadrature. The homogeneous solution $\overline{u}_{12h}^{(0)}$ turns out to be

$$\overline{u}_{12h}^{(0)} = a\tilde{y} + b. \tag{3.75}$$

The constant a should vanish because of the matching with the outer solution while the value of b is obtained by forcing (3.74). The spatial distribution of $\overline{u}_{12p}^{(0)}$ along with the value of b are the relevant results of the present analysis since they provide the structure of the steady drift induced by a rippled bed in the boundary layer, its asymptotic behaviour for $\tilde{y} \to \infty$ and the amount of vorticity which diffuses and is advected in the entire water column.

At this stage it is useful to stress that the flow in the bottom boundary layer exactly matches that in the core region. Indeed for $\tilde{y}$ tending to infinity u_{00} matches U_0 while $u_{01}, v_{01}, u_{02}, v_{02}$ tend to zero. Moreover v_{10} has a term proportional to y^* which matches V_0 and a constant part which matches $V_{1p}^{(12)}$. The component $\hat{u}_{10}$ for large $\tilde{y}$ fits $U_{1p}^{(2)}$ while u_{11}, v_{11} tend to vanish. Finally, $u_{12}^{(0)}$ and $v_{12}^{(0)}$ for $\tilde{y}$ tending to infinity give rise to time-periodic functions of order $\delta\epsilon^2$ which force time-periodic components of order ϵ^2 in U_{1s} and V_{1s}. In particular the time average of $u_{12}^{(0)}$ for large values of $\tilde{y}$ gives rise to an $O(\epsilon^2)$ correction to the boundary condition for the steady mass flux ($\overline{U}_{1s}$) provided by the limit of $\overline{u}_{10}$ for $\tilde{y}$ tending to infinity.

4. The results

From the analysis previously described, the flow in the boundary layer at the bottom of a sea wave when ripples cover the bottom surface can be seen as the sum of different contributions. There is a part of $O(1)$ (u_{00}) which is periodic in time, slowly varying in the direction of wave propagation and with a vertical structure described by the well-known Stokes' solution. The bottom waviness induces a perturbation to the flow of $O(\epsilon)$ [(u_{01}, v_{01})], the main characteristic of which is the presence of

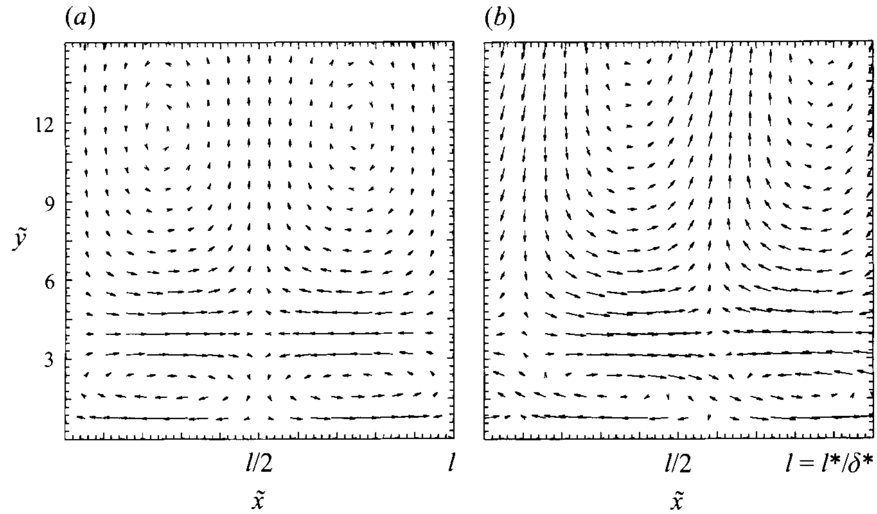

FIGURE 1. Steady part of $[(\hat{u}_{01}, \hat{v}_{01}) + \delta(\hat{u}_{11}, \hat{v}_{11})]e^{i\alpha\tilde{x}} + \text{c.c.}$ for $\alpha = 0.125, Re = 1$. (a) $\delta = 0$, (b) $\delta = 0.001$.

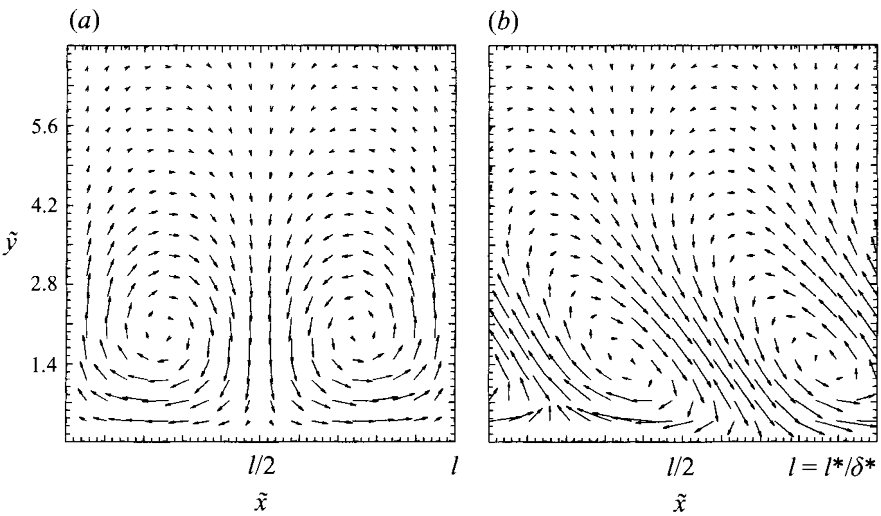

FIGURE 2. Steady part of $[(\hat{u}_{01}, \hat{v}_{01}) + \delta(\hat{u}_{11}, \hat{v}_{11})]e^{i\alpha\tilde{x}} + \text{c.c.}$ for $\alpha = 0.875, Re = 1$. (a) $\delta = 0$, (b) $\delta = 0.05$.

steady recirculting cells periodic in the x-direction and symmetric with respect to the ripple crests (see figures 1*a* and 2*a*). The number, form and intensity of these cells depend on the wavenumber α of the waviness and the flow Reynolds number Re. The reader interested in a discussion of the behaviour of u_{01}, v_{01} as α and Re are varied is referred to Vittori (1989). However no steady mass flux is induced in the $\tilde{x}$-direction. The steady drift also vanishes at order ϵ^2. Indeed u_{02} and v_{02} have a component periodic in the $\tilde{x}$-direction and a component which does not depend on $\tilde{x}$. However the latter has a vanishing time-average because of the symmetry of the problem along the x-direction at this order of approximation. At order δ, the mean dynamic pressure field and the effect of the time averaged convective terms give rise to a mean streaming current $\overline{u}_{10}$ which was first obtained by Longuet-Higgins (1953).

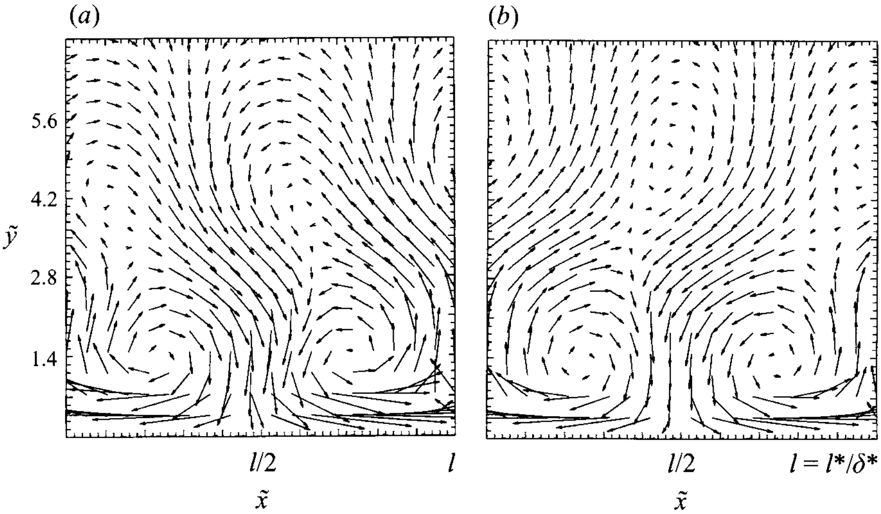

FIGURE 3. Steady part of $[(\hat{u}_{01}, \hat{v}_{01}) + \delta(\hat{u}_{11}, \hat{v}_{11})]e^{i\alpha\tilde{x}}$ + c.c. for $\alpha = 0.5$. (*a*) *Re*=50, $\delta = 0.001$, (*b*) *Re*=90, $\delta = 0.0002$.

Moreover oscillating contributions are present $[(\tilde{u}_{10}, \tilde{v}_{10})]$. At order $\delta\epsilon$ the interaction of the flow described by Vittori (1989), which is periodic both in space and in time, with the Stokes' flow and that obtained by Longuet-Higgins (1953) forces velocity components $[(u_{11}, v_{11})]$ which are periodic in the $\tilde{x}$-direction. Therefore at this order no steady mass flux is induced. However u_{11} and v_{11} are no longer symmetric with respect to the ripple crests and produce a distortion of the form of the recirculating cells appearing at order ϵ. Finally, a steady streaming uniform in the x-direction is induced by the bottom waviness at order $\delta\epsilon^2$. Such steady drift is forced by: (i) the interaction of the mean streaming current of order δ with the component of the flow at order ϵ^2 which is independent of $\tilde{x}$ (last term of (3.73)); (ii) the interaction of the spatial periodic components of order ϵ and $\delta\epsilon$ (first and second terms on the right-hand side of (3.73)); (iii) the interaction of the time-periodic Stokes' flow with the vertical component of the flow at order $\delta\epsilon^2$, which turns out to be time-dependent too (third term appearing on the right-hand side of (3.73)).

In the description of the results, we will focus our attention on the flows at order $\epsilon\delta$ and $\epsilon^2\delta$ which are the novelty of the present work.

In figures 1 and 2, the steady part of $[(\hat{u}_{01}, \hat{v}_{01}) + \delta(\hat{u}_{11}, \hat{v}_{11})]e^{i\alpha\tilde{x}}$ + c.c. is plotted for fixed values of α and *Re* (namely $\alpha = 0.125, Re = 1$ and $\alpha = 0.875, Re = 1$ respectively) both for $\delta = 0$ (figures 1*a* and 2*a*) and for non-vanishing values of δ ($\delta = 0.001$ in figure 1*b* and $\delta = 0.05$ in figure 2*b*) to show the effect of the Stokes' drift on the recirculating cells described by Vittori (1989) who neglected $O(\delta)$ effects. According to figure 8 of Vittori's (1989) paper, when $\delta = 0$ there are four steady recirculating cells for $\alpha = 0.125$ while for $\alpha = 0.875$ only two cells appear. In both cases the presence of the Stokes drift causes a shift of the centres of the cells which is in the offshore direction (we remind the reader that the x-axis points offshore). Moreover a distorsion of the steady streaming is also induced. As pointed out in §2, results for values of *Re* falling in the range characteristic of the laminar regime can be obtained (*Re* up to 100). For example figure 3 shows $[(\hat{u}_{01}, \hat{v}_{01}) + \delta(\hat{u}_{11}, \hat{v}_{11})]e^{i\alpha\tilde{x}}$ + c.c for $\alpha = 0.5$ and values of *Re* equal to 50 and 90. Also for these high values of *Re*, the spatial periodic steady streaming is no longer symmetric with respect to the crests and the troughs

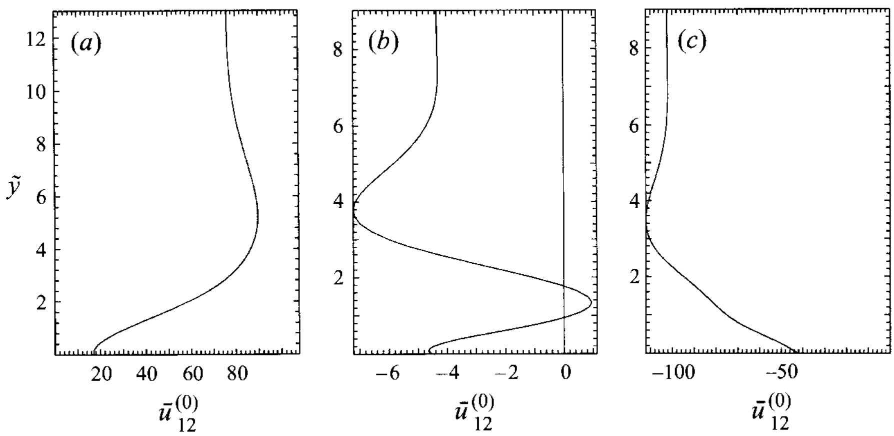

FIGURE 4. Plot of the velocity component $\bar{u}_{12}^{(0)}$ versus $\tilde{y}$ for (*a*) $\alpha = 0.3$, $Re = 12$, (*b*) $\alpha = 0.4$, $Re = 6$, (*c*) $\alpha = 1$, $Re = 10$.

of the ripples and is also distorted. As it will be described in a forthcoming paper (Blondeaux *et al.* 1996) this distortion is quite important in understanding sediment transport in the direction of wave propagation, since it induces the migration of the bottom waviness (i.e. ripple migration) when the non-cohesive bottom case is considered.

Figure 4 shows some typical examples of the vertical structure of the steady uniform flow $\bar{u}_{12}^{(0)}$ which is induced at order $\delta\epsilon^2$. There are values of α, Re such that the correction of the steady drift is always negative and values of α, Re such that it turns out to be always positive. In other words, ripple presence increases mass transport towards the shore in the former case and reduces it in the latter case. Moreover values of α, Re exist for which $\bar{u}_{12}^{(0)}(y)$ is characterized by a twisted behaviour, such that the mass transport is negative close to the bed, positive moving far from it and eventually becomes negative again as $\tilde{y}$ tends to infinity. However these are uncommon cases and ripple influence on mass transport can be discussed by looking at the asymptotic value $\bar{u}_M$ of $\bar{u}_{12}^{(0)}$ for large $\tilde{y}$. The values of $\bar{u}_M$ are of importance for the determination of the steady flow in the core region where y^* is of order h^*. Indeed the limit of $\bar{u}_{12}^{(0)}$ for $\tilde{y}$ tending to infinity is just the boundary condition for $\overline{U}_{1s}$ at y equal to zero and thus is fundamental to obtaining the mass transport in the entire water column. An order of magnitude analysis on the equations of motion shows that in the present case, even though a_o^* is assumed to be of the same order of magnitude as δ^*, the solution in the core region can be obtained by the creeping flow approximation. Since this case is quite simple to analyse, no further attention will be paid to it.

For small values of Re, $\bar{u}_M$ turns out to be always negative in the range of α investigated here (see figure 5). The waviness of the bed thus increases mass transport towards the shore. When Re is increased, it is hard to single out an overall tendency, as shown in figure 6. Indeed for $\alpha = 1$, $\bar{u}_M$ is always negative and its absolute value increases as Re is increased, at least in the range of Re investigated here. However when smaller values of α are considered an increasingly oscillatory behaviour appears. For example when α is equal to 0.3 and small values of Re are considered $\bar{u}_M$ is negative. Then increasing Re, $\bar{u}_M$ becomes positive but further increases of Re lead to negative values again. Moreover the behaviour of the function

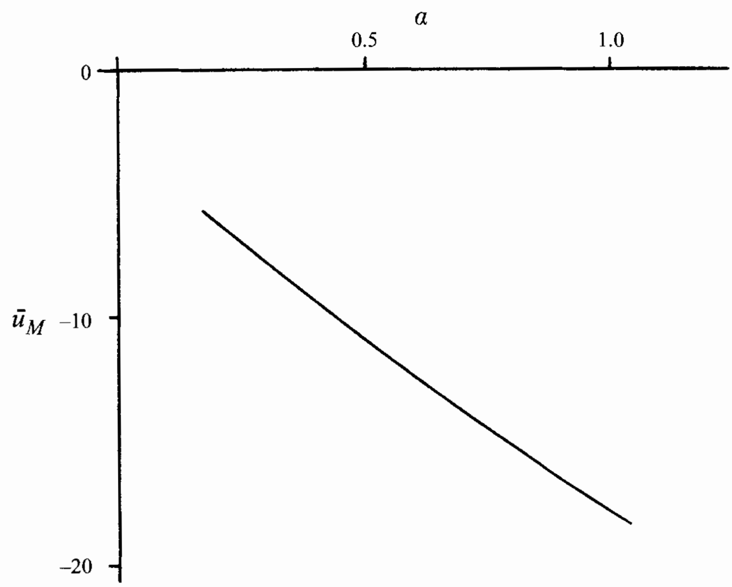

FIGURE 5. Asymptotic values $\overline{u}_M$ of $\overline{u}_{12}^{(0)}$ for large values of $\tilde{y}$ plotted versus α for $Re = 1$.

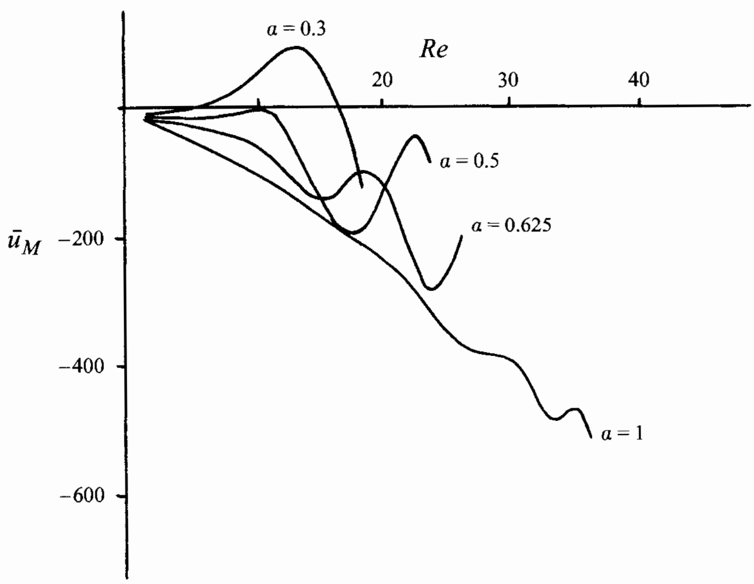

FIGURE 6. Asymptotic values $\overline{u}_M$ of $\overline{u}_{12}^{(0)}$ for large values of $\tilde{y}$ plotted versus Re and different values of α.

$\overline{u}_M(Re)$ for larger α seems to indicate that for $\alpha = 0.3$ increasing Re, positive values of $\overline{u}_M$ could be found again and so on. Because of the difficulties in finding a sufficiently accurate solution for large Re and small α, the curves appearing in figure 6 have been interrupted. Indeed the numerical quadrature of (3.73) leads to values of $\overline{u}_{12}^{(0)}$ which are much smaller than the right-hand side of (3.73). It follows that the numerical accuracy which is required in the evaluation of $\mathscr{F}(y)$ should be quite high. When Re is larger than a critical value which depends on α, it turns out that it is not possible to find an accurate estimate of $\overline{u}_{12}^{(0)}$ and hence of $\overline{u}_M$ even using double precision in the computer codes. In these cases an asymptotic solution for large values of Re and $\alpha \sim O(Re^{-1})$ could be of some help. Figure 6 gives an idea of the above critical values of Re.

When a non-cohesive bed is considered, the changeability of the sea bottom introduces further degrees of freedom and makes it difficult to state the influence of mass transport on the process of ripple formation and development. Such influence can be determined by introducing $O(\delta)$ effects in the analyses by Blondeaux (1990) and Vittori & Blondeaux (1990). This work has been accomplished in a companion paper (Blondeaux *et al.* (1996)) to be submitted for publication.

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Appendix

To solve equation (3.64) along with boundary conditions (3.66)–(3.68) a new coordinate system (X, Y) which oscillates with the fluid far from the bottom is introduced:

$$X = \tilde{x} + \frac{Re}{4}\int_0^{\tilde{t}} (e^{i\tilde{t}} + \text{c.c.})\,d\tilde{t}, \tag{A 1}$$

$$Y = \tilde{y} \tag{A 2}$$

and a modified velocity field of order one is defined:

$$\mathscr{U}_{00} = u_{00} + \tfrac{1}{2}(e^{i\tilde{t}} + \text{c.c.}). \tag{A 3}$$

Finally a new unknown is introduced:

$$\mathscr{V}_{11} = \hat{v}_{11} P(\tilde{t}) \tag{A 4}$$

where

$$P(t) = \exp\left[-\tfrac{1}{4} i\alpha Re \int_0^{\tilde{t}} (e^{i\tilde{t}} + \text{c.c.})\,d\tilde{t}\right]. \tag{A 5}$$

By taking into account that both $\hat{u}_{01}$ and $\hat{v}_{01}$ can be expressed in a fashion similar to (A4),

$$(\mathscr{U}_{01}, \mathscr{V}_{01}) = (\hat{u}_{01}, \hat{v}_{01}) P(\tilde{t}), \tag{A 6}$$

and by expanding both $\mathscr{V}_{11}$ and $P(t)$ in Fourier series,

$$P(\tilde{t}) = \sum_{m=\infty}^{\infty} p_m e^{im\tilde{t}}, \quad \hat{\mathscr{V}}_{11} = \sum_{m=-\infty}^{\infty} G_m(Y) e^{im\tilde{t}}, \tag{A 7}$$

a system of coupled ordinary differential equations for G_m is obtained from (3.64):

$$-\tfrac{1}{2}N^4 G_m + imN^2 G_m + \frac{i\alpha Re}{2}\left[F_o N^2 G_{m-1} + (F_o)^{\dagger} G_{m+1} - \frac{d^2 F_o}{dY^2} G_{m-1} - \left(\frac{d^2 F_o}{dY^2}\right)^{\dagger} G_{m+1}\right] = H_m \tag{A 8}$$

with the following boundary conditions:

$$G_m = -\frac{\mathrm{d}\hat{v}_{10}}{\mathrm{d}y} p_{m-1} + \left(\frac{\mathrm{d}\hat{v}_{10}}{\mathrm{d}y}\right)^{\dagger} p_{m+1} \quad \text{for } \tilde{y} = 0, \tag{A 9}$$

$$\frac{\mathrm{d}G_m}{\mathrm{d}Y} = -2\pi \left[imu_{01}^{(m)} + \frac{i\alpha Re}{4}\left(u_{01}^{(m-1)} + u_{01}^{(m+1)}\right)\right] + i\alpha \left[\left(\frac{\mathrm{d}\bar{u}_{10}}{\mathrm{d}\tilde{y}} + \text{c.c.}\right) p_m + \frac{\mathrm{d}\hat{u}_{10}}{\mathrm{d}y} p_{m-2} + \left(\frac{\mathrm{d}\hat{u}_{10}}{\mathrm{d}y}\right)^{\dagger} p_{m+2}\right] \quad \text{for } \tilde{y} = 0. \tag{A 10}$$

In (A8) the operator N^2 is defined as follows:

$$N^2 \equiv \frac{\mathrm{d}^2}{\mathrm{d}Y^2} - \alpha^2 \tag{A 11}$$

and $H_m(Y)$ are the coefficients of the Fourier expansion of the forcing term appearing in (3.64) in the new reference frame. Moreover the function F_0 is defined as

$$F_0(Y) = \tfrac{1}{2}\mathrm{e}^{-(1+\mathrm{i})Y}. \tag{A 12}$$

Neglecting harmonics higher than the Mth in the Fourier series (A7), the functions G_m are determined numerically using a Runge–Kutta method of fourth order and a shooting procedure from some large value Y_{start} of Y. More details can be found in Vittori & Blondeaux (1990).

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